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UNIONS OF GROUPS OF SMALL HEIGHT

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A semigroup S is called a union of groups if each of its elements lies in a (maximal) subgroup of S. It is well known that in a union of groups Green's relation $\mathscr{J}$ (defined by $a\mathscr{J}b$ if and only if $SaS=SbS$) is a congruence whose classes are completely simple subsemigroups of S and such that $S/\mathscr{J}$ is a semilattice (i.e. satisfies $x^2=x$, $xy=yx$) [1].

In this paper we present a construction of all unions of groups S having the following properties:

(1) S is the syntactic semigroup of a language of the form $C^+=\bigcup_{n>0}C^n$, where C is a finite prefix code;

(2) $S/\mathscr{J}$ is the two-element semilattice.

Using the terminology of Clifford and Preston the semigroups studied here are ideal extensions of a completely simple semigroup by another, or unions of groups of height two. All the completely simple semigroups satisfying condition (1) above have been obtained in [6], using the concept of team tournament. All unions of groups S satisfying (1) and having a non-trivial group of units are constructible using certain factorizations of $\mathbb{Z}_n$, the group of integers modulo n [4]. The two construction techniques are combined here to give all the unions of groups satisfying (1) and (2) as transition semigroups of what we call a 'standard amalgamation' of the automata of two team tournaments (Theorem 3.2).

It is likely that the process given here generalizes to a construction of all unions of groups satisfying (1) provided some convenient representation of the two $\mathscr{J}$-classes case is found. We conjecture that all these semigroups are chains of length n of completely simple semigroups, and that they are – in general – of group complexity n, in the sense of J. Rhodes [3] (cf. also Tilson's Chapter 12 in [2]). The complexity conjecture, at least in the case $n=2$, can be verified on examples using the results of [11].

Directly related to these considerations is the problem of describing the variety $\mathscr{V}$ of all languages whose syntactic semigroups are unions of groups (see [2], [10]). In view of the recent results of J.E. Pin [9], the following question arises naturally: Is

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$\mathscr{V}$ generated by its finite prefix codes (i.e. is it true that $\mathscr{V}$ is the smallest variety containing all the languages C^+ where C is a finite prefix code, and the syntactic semigroup of C^+ is a union of groups)? Our results are the initial steps toward an answer.

The first section of this paper contains preliminary results on factorizations of $\mathbb{Z}_n$ and a determination of certain permutations of $\mathbb{Z}_n$ preserving a transversality property. In Section 2 we define the notion of amalgamation of automata. This is a simple construction creating an automaton on the disjoint union of the sets of states of others, merging all initial states into a single one. Section 3 contains our main result: The unions of groups satisfying (1) and (2) are the transition semigroups of the automaton obtained by an appropriate amalgamation of automata of team tournaments. The proofs are presented in Section 4.

We recall that given a language L in A^+ (i.e. a subset of the free semigroup A^+ on the set A), the syntactic semigroup Synt(L) is defined as the quotient of A^+ by the congruence $\Sigma(L)=\{(u,v)\in A^+\times A^+ : xuy\in L \Leftrightarrow xvy\in L, \text{ for every } x,y\in A^*\}$. For other undefined terminology, we refer the reader to [1], [2], or [5].

1. Perfect transversals for certain factorizations of $\mathbb{Z}_n$

An equivalence relation ϱ on $\mathbb{Z}_n$ is called a *perfect partition* if ϱ admits a system of representatives T such that $T+i$ (modulo n) remains a system of representatives of ϱ for every i. The set T is then called a *perfect transversal*, and one can show that the class of 0 modulo ϱ, say K, together with T form a factorization of $\mathbb{Z}_n$ in the following sense: Every $x\in\mathbb{Z}_n$ can be written uniquely $x=k+t$ with $k\in K$, $t\in T$ (see [4]). Since the problem of finding all factorizations of $\mathbb{Z}_n$ is open, the same holds for finding all the possible perfect transversals of $\mathbb{Z}_n$. However, among the factorizations of $\mathbb{Z}_n$ there are factorizations $\mathbb{Z}_n=K\oplus T$ that do not require any reduction modulo n. They are called factorizations of the set $\{0,1,\ldots,n-1\}$, and are all obtainable as follows:

Let $k_1\,|\,k_2\,|\cdots|\,k_N\,|\,n$ be a chain of divisors of n ($|$ reads 'divides', and $k_1<k_2<\cdots<k_N<n$ with k_1 possibly being 1). Form the polynomials

$$p(x)=\frac{1-x^{k_1}}{1-x}\,\frac{1-x^{k_3}}{1-x^{k_2}}\cdots \quad\text{and}\quad q(x)=\frac{1-x^{k_2}}{1-x^{k_1}}\,\frac{1-x^{k_4}}{1-x^{k_3}}\cdots.$$

Then $(1-x^n)/(1-x)=p(x)q(x)$, and $\{0,1,\ldots,n-1\}=K\oplus T$ where K [resp. T] is the set of exponents of the terms of $p(x)$ [resp. $q(x)$]. Furthermore, since

$$\frac{1-x^n}{1-x}\equiv p(x)[q(x)x^i] \quad \text{modulo } (1-x^n),$$

T is a perfect transversal of the partition π whose classes are the various subsets $K+t$, $t\in T$. We shall say that π is the perfect partition defined by the sequence $k_1\,|\,k_2\,|\cdots|\,k_N\,|\,n$. Besides the *basic* transversal T defined above, we propose to find all the possible perfect transversals of π.

Proposition 1.1. *Any perfect transversal T of the partition of $\mathbb{Z}_n$ defined by the sequence $k_1 \mid k_2 \mid \cdots \mid k_N \mid n$ is obtainable from a perfect transversal T' of the partition of $\mathbb{Z}_{k_N}$ defined by the sequence $k_1 \mid k_2 \mid \cdots \mid k_{N-1} \mid k_N$ as follows:*

(a) *In case N is even, $T = \{t' + \lambda k_N : t' \in T' \text{ and } \lambda \text{ arbitrary}\}$;*

(b) *In case N is odd, $T = \bigcup_{\lambda \geq 0} (T' + \lambda k_N)$.*

Proof. We denote by π [resp. π'] the partition of $\mathbb{Z}_n$ [resp. $\mathbb{Z}_{k_N}$] defined by the sequence $k_1 \mid k_2 \mid \cdots \mid k_N \mid n$ [resp. $k_1 \mid k_2 \mid \cdots \mid k_N$].

In case N is even, each class of π is obtained from a class of π' by addition of $k_N, 2k_N, \ldots, n - k_N$. Diagram 1, where the rows are the different classes of π, shows the relationship between π and π'.

$$
\begin{array}{|cccc|}
\hline
0 & 1 & 2 \cdots k_1 - 1 \cdots & (k_{N-1} - k_{N-2} + \cdots + k_1 - 1) \\
k_1 & k_1 + 1 & & \\
\vdots & & & \vdots \\
(k_N - k_{N-1} + \cdots + k_2 - k_1) & & & k_{N-1} \\
\hline
\end{array}
$$

$$
\begin{array}{|cc|}
\hline
k_N \cdots & (n - k_N + k_{N-1} - \cdots + k_1 - 1) \\
 & \\
\cdots & n - 1 \\
\hline
\end{array}
$$

Diagram 1

Since N is even each class of π is globally invariant under the addition of k_N (modulo n). Hence every perfect transversal T of π gives by reduction modulo k_N a perfect transversal T' of π'. Conversely, when adding to each element t' of a perfect transversal T' of π' an arbitrary multiple of k_N we obtain a transversal T of π. For every i, the elements of $T + i$ are distributed in the same classes as their residues modulo k_N, that is as the elements of $T' + i$. Since T' is a perfect transversal, the same holds true for T.

In case N is odd, the classes of π are those of π' and their translates by multiples of k_N as indicated by the rows of Diagram 2. Reduction modulo k_n shows that any perfect transversal T of π induces a perfect transversal T' of the partition π'. Also T induces transversals on each classes of the blocks $[k_N, 2k_N - 1]$, $[2k_N, 3k_N - 1], \ldots,$ $[n - k_N, n - 1]$ that are perfect when reduced modulo k_N. The assertion in (b) is that these transversals are precisely $T' + k_N$, $T' + 2k_N, \ldots, T' + n - k_N$.

To prove it we may assume (in view of the fact that T is perfect) that $0 \in T$ and show that this implies $k_N, 2k_N, \ldots,, n - k_N \in T$. Proceeding by induction on N, we consider the set S of all multiplies of k_2. The partition π defines a partition π_S on S

$$
\begin{array}{cccc}
0 & 1 \quad 2 \cdots k_1 - 1 & \cdots & (k_N - k_{N-1} + \cdots + k_1 - 1) \\
k_1 & & & \\
\vdots & & & \\
(k_{N-1} - k_{N-2} + \cdots + k_2 - 1) & & \cdots & k_N - 1 \\
\hdashline
k_N & & & \\
k_N + k_1 & & & \\
\vdots & & & \\
(n - k_N + k_{N-1} - \cdots + k_2 - k_1) & & \cdots & n - 1
\end{array}
$$

Diagram 2

admitting a transversal $T \cap S$ which remains a transversal when adding multiples of k_2. Hence the classes of π_S are obtained from a partition π'' defined by the sequence

$$\frac{k_3}{k_2} \Big| \cdots \Big| \frac{k_N}{k_2} \Big| \frac{n}{k_2}$$

by multiplying all elements by k_2, and $T \cap S$ is also obtained from a perfect transversal T'' of this partition by multiplication by k_2. By the induction hypothesis T'' contains

$$\frac{k_N}{k_2}, \frac{2k_N}{k_2}, \ldots, \frac{n - k_N}{k_2},$$

hence $k_N, 2k_N, \ldots, n - k_N \in T$.

Example. The sequence $3|6|12|36$ defines the following partition of $\mathbb{Z}_{36}$:

$$
\begin{array}{cccccc}
\underline{0} & 1 & 2 & 6 & 7 & 8 \\
\underline{3} & 4 & 5 & \underline{\underline{9}} & 10 & 11 \\
\underline{12} & 13 & 14 & 18 & 19 & 20 \\
\underline{15} & 16 & 17 & \underline{\underline{21}} & 22 & 23 \\
\underline{24} & 25 & 26 & 30 & 31 & 32 \\
\underline{27} & 28 & 29 & \underline{\underline{33}} & 34 & 35
\end{array}
$$

There are exactly 2 perfect transversals containing 0: $\{0, 3, 12, 15, 24, 27\}$ and $\{0, 9, 12, 21, 24, 33\}$.

Letting T be the basic perfect transversal of the partition of $\mathbb{Z}_n$ defined by the sequence $k_1 | k_2 | \cdots | k_N | n$, we propose to find all the permutations φ of $\mathbb{Z}_n$ such that $\varphi(T + i)$ is a perfect transversal for every $i = 0, 1, \ldots$. We call such a permutation φ a *perfect permutation* corresponding to $k_1 | k_2 | \cdots | k_N | n$.

We denote by $\mathscr{S}_k$ the symmetric group on the set $\{0, 1, \ldots, k-1\}$.

Proposition 1.2. *Every perfect permutation $\hat{\varphi}$ of $\mathbb{Z}_n$ corresponding to the sequence $k_1 | k_2 | \cdots | k_N | n$ is obtainable by extending a perfect permutation φ of $\mathbb{Z}_{k_N}$ corresponding to the sequence $k_1 | k_2 | \cdots | k_{N-1} | k_N$ as follows:*

For every $i = 0, 1, \ldots, k_N - 1$ choose a permutation $\sigma_i \in \mathcal{S}_{n/k_N}$ and define $\hat{\varphi}$ by $\hat{\varphi}(j) = \varphi(i) + \sigma_i(\lambda)k_N$ where λ and i are the quotient and the remainder of the division of j by k_N, $j = i + \lambda k_N$ with $0 \le i < k_N$.

Proof. $\hat{\varphi}(j) = \hat{\varphi}(j')$ implies $\varphi(i) + \sigma_i(\lambda)k_N = \varphi(i') + \sigma_{i'}(\lambda')k_N$ with an obvious notation. Reduction modulo k_N gives $\varphi(i) = \varphi(i')$, hence $i = i'$, and $\lambda = \lambda'$. This shows that $\hat{\varphi}$ is a bijection of $\mathbb{Z}_n$. Constructing any perfect transversal T of the partition of $\mathbb{Z}_n$ from a perfect transversal T' of $\mathbb{Z}_{k_N}$ by the formula (a) or (b) of Proposition 1.1 a computation of $\hat{\varphi}(T+s)$ shows that it is a perfect transversal of the partition of $\mathbb{Z}_n$ for every $s = 0, 1, \ldots, n-1$. Conversely assume that $\hat{\varphi}$ is a perfect permutation of $\mathbb{Z}_n$. We consider successively the cases N even and N odd.

N even. Every perfect transversal of $\mathbb{Z}_n$ is obtained from a perfect transversal of $\mathbb{Z}_{k_N}$ by adding arbitrary multiples of k_N. With T the basic transversal, we observe that in the transversals $T, T + k_{N-1}, T + 2k_{N-1}, \ldots, T + (n - k_{N-1})$, any two consecutive transversals have a segment of length at least k_{N-1} in common. Hence by Proposition 1.1(b) the images of these transversals under $\hat{\varphi}$ are the same modulo k_N. In particular $\hat{\varphi}(T), \hat{\varphi}(T + k_N), \hat{\varphi}(T + 2k_N), \ldots, \hat{\varphi}(T + n - k_N)$ are congruent transversals modulo k_N and for every $i \in T, \hat{\varphi}(i), \hat{\varphi}(i + k_N), \ldots, \hat{\varphi}(i + n - k_N)$ differ by multiples of k_N. Since $\hat{\varphi}$ is 1–1 there exists a permutation σ_i of $\{0, 1, \ldots, n/k_N\}$ such that $\hat{\varphi}(i + \lambda k_N) = \hat{\varphi}(i) + \sigma_i(\lambda)k_N$. A similar argument applies to the successive transversals $T+1, T+2, \ldots, T + k_{N-1} - k_{N-2} + \cdots + k_1 - 1$, showing that the restriction of $\hat{\varphi}$ to $[0, k_N - 1]$ defines a perfect permutation φ of $\mathbb{Z}_{k_N}$ such that $\varphi(i) \equiv \hat{\varphi}(i)$ modulo k_N. In the formula above giving $\hat{\varphi}(i + \lambda k_N)$, we may eventually modify the permutation σ_i so that $\hat{\varphi}(i + \lambda k_N) = \varphi(i) + \sigma_i(\lambda)k_N$.

N odd. In the basic transversal $T = \{0, k_1, 2k_1, \ldots, n - k_N + \cdots + k_2 - k_1\}$ we consider the following intervals:

$$I_0 = [0, k_1, \ldots, k_{N-3} - k_{N-4} + \cdots + k_2 - k_1],$$

$$I_1 = [k_N, k_N + k_1, \ldots, k_N + k_{N-3} - k_{N-4} + \cdots + k_2 - k_1],$$

$$I_{(n/k_N)-1} = [n - k_N, n - k_N + k_1, \ldots, n - k_N + k_{N-3} - k_{N-4} + \cdots + k_2 - k_1].$$

These are n/k_N intervals of T of a certain length l (in fact, $l = (k_{N-3}/k_{N-4}) \cdots (k_4/k_3)(k_2/k_1))$, extracted from a subdivision of T into intervals of equal length l. For example, between I_0 and I_1 there is a gap consisting of $I_0 + k_{N-2}, I_0 + 2k_{N-2}, \ldots, I_0 + k_{N-1} - k_{N-2}$. Forming successively the transversals $T + k_N - k_{N-1} + k_{N-2}$, $T + k_N - k_{N-1} + 2k_{N-2}, \ldots, T + k_N - k_{N-1} + (k_{N-1} - k_{N-2}) = T + k_N - k_{N-2}, T + k_N = T$, we obtain k_{N-1}/k_{N-2} distinct perfect transversals having the n/k_N intervals above in common. For example, $(I_0 + k_{N-1} - k_{N-2}) + k_N - k_{N-1} + k_{N_2} = I_0 + k_N = I_1$. It follows that $J = \hat{\varphi}(I_0 \cup I_1 \cup \cdots \cup I_{n/k_N - 1})$ consists of $l \cdot n/k_N$ elements belonging to k_{N-1}/k_{N-2} distinct perfect transversals of the partition of $\mathbb{Z}_n$. However I_0 itself appears in exactly k_{N-1}/k_{N-2} distinct transversals of the corresponding partition of $\mathbb{Z}_{k_N}$. Hence by (b) of Proposition 1.1, J consists of l representatives of the classes of the partition of $\mathbb{Z}_{k_N}$ together with all their translates by multiples of k_N. Similarly all the intervals $I_0+\mu k_{N-2}, I_1+\mu k_{N-2}, \ldots, I_{(n/k_N)-1}+\mu k_{N-2}$ are globally mapped by $\hat{\sigma}$ onto sets that are invariant under addition of multiples of k_N. Thus $\hat{\varphi}$ defines a perfect permutation φ (after reduction modulo k_N) of each set $[0, k_N-1]$, $[k_N, 2k_N-1], \ldots, [n-k_N, n-1]$. It remains to show that φ is the same on each of these intervals. Proceeding by induction on N, the permutation φ on $[0, k_N-1]$ maps the set of all multiples of k_2 onto a set of elements that are congruent modulo k_2. Since any perfect transversal of the partition of $\mathbb{Z}_n$ is determined by its restriction to $[0, k_N-1]$, $\hat{\varphi}$ has the same property than φ with respect to the multiples of k_2. Hence we may assume that $\hat{\varphi}$ maps the set of all multiples of k_2 onto themselves, in such a way that if $T'=\{0, k_3, 2k_3, \ldots, n-k_3\}$, then $\hat{\varphi}(T'+sk_2)$ is a perfect transversal for all s. Thus $\hat{\varphi}$ defines a perfect permutation corresponding to the sequence

$$\frac{k_3}{k_2} \,\Big|\, \frac{k_4}{k_2} \,\Big|\, \cdots \,\Big|\, \frac{k_N}{k_2} \,\Big|\, \frac{n}{k_2}.$$

By the induction hypothesis for any two multiples i, j of $k_2, i \equiv j$ modulo k_N implies $\hat{\varphi}(i) \equiv \hat{\varphi}(j)$ modulo k_N, and the same holds true for any pair i, j such that $i \equiv j$ modulo k_2. This shows that $\hat{\varphi}$ defines the same perfect permutation on each set $[0, k_N-1], [k_N, 2k_N-1], \ldots, [n-k_N, n-1]$ and completes the proof of Proposition 1.2.

The *basic modulus* of the sequence $k_1 | k_2 | \cdots | k_N | n$ is defined to be k_1 if $k_1 \neq 1$ and k_2 otherwise.

Corollary 1.3. *Let $\hat{\varphi}$ be a perfect permutation of $\mathbb{Z}_n$ corresponding to the sequence $k_1 | k_2 | \cdots | k_N | n$ of basic modulus k. The following conditions on $\hat{\varphi}$ are equivalent:*

(1) *$\hat{\varphi}(0)=0$ and $\hat{\varphi}(i) \geq i-1$ for every $i, 1 \leq i < n$,*

(2) *$\hat{\varphi}$ induces a permutation $\sigma \in \mathscr{S}_k$ such that $\sigma(0)=0$, $\sigma(i) \geq i-1$, and $\hat{\varphi}(i+\lambda k)=\sigma(i)+\lambda k$ for every $i, 0 \leq i < k$, and for every λ, $0 \leq \lambda < n/k$.*

Proof. It is clear that (2) implies (1). Assume that $\hat{\varphi}$ satisfies (1). Following the notation of Proposition 1.2 we have $\hat{\varphi}(i+\lambda k_N)=\varphi(i)+\sigma_i(\lambda)k_N$ for every $i= 0, 1, \ldots, k_N-1$, $\lambda=0, 1, \ldots, n/k_N-1$, and φ a perfect permutation of $\mathbb{Z}_{k_N}$. Consequently

$$i+\lambda k_N-1 \leq \hat{\varphi}(i+\lambda k_N) \leq k_N-1+\sigma_i(\lambda)k_N. \tag{1.3.1}$$

It follows that $[\lambda-\sigma_i(\lambda)]k_N \leq k_N-i$. In case $i>0$, this implies $[\lambda-\sigma_i(\lambda)]k_N < k_N$, hence $\lambda-\sigma_i(\lambda) \leq 0$, or $\lambda \leq \sigma_i(\lambda)$, forcing $\lambda=\sigma_i(\lambda)$ for every λ. In case $i=0$, we obtain similarly $\sigma_0(\lambda) \geq \lambda-1$. However, $\hat{\varphi}(0)=0$ implies $\varphi(0)=0$ and $\sigma_0(0)=0$. Hence for $i=0$ the first inequality (1.3.1) becomes $\lambda k_N-1 \leq \sigma_0(\lambda)k_N$. In case $k_N \neq 1$ this gives $(\lambda-1)k_N < \lambda k_N-1 \leq \sigma_0(\lambda)k_N$, and thus $\sigma_0(\lambda) > \lambda-1$. Again this forces $\sigma_0(\lambda)=\lambda$ for every λ. Consequently $\hat{\varphi}$ restricted to $[0, \ldots, k_N-1]$ is φ, a perfect permutation of

$\mathbb{Z}_{k_N}$ satisfying the conditions (1). By induction on N, (2) follows from the fact that the perfect permutations of $\mathbb{Z}_k$ corresponding to the sequences k or $1\,|\,k$ are all the permutations of $0, 1, \ldots, k-1$ and for these, (1) and (2) are obviously equivalent.

2. Amalgamation of automata

Given a collection of pairwise disjoint non empty sets S_i $(i \in I)$ with a distinguished element s_i^0 in each set S_i, we call *amalgamated sum* of the sets S_i the set $S = *_{i\in I}(S_i, s_i^0)$ defined by $S = [\bigcup_{i\in I}(S_i - \{s_i^0\})] \cup \{s_0\}$ where s_0 denotes an element not in any of the sets S_i $(i \in I)$. Thus the amalgamated sum is simply the set obtained by forming the union and merging the distinguished elements in each set into a single element denoted by s_0.

Let $\mathfrak{A}_i = (S_i, f_i)$, $i = 1, 2, \ldots, n$, be a family of finite state A_i^+-automata with pairwise disjoint sets of states S_i and input alphabets A_i, the functions $f_i : S_i \times A_i \to S_i$ being the usual transition functions. We assume that in each set S_i and initial state s_i^0 has been distinguished, and that we are given a collection of functions $\varphi_{ij} : S_i \times A_j \to S_j$ such that

(1) $\varphi_{ii} = f_i$;

(2) $\varphi_{ij}(s_i^0, a_j) = f_j(s_j^0, a_j)$ for every $i, j = 1, 2, \ldots, n$, and for every $a_j \in A_j$.

We define the amalgamation $*_{i=1}^n(\mathfrak{A}_i, \varphi_{ij})$ of the automata $\mathfrak{A}_i$ as the automaton $\mathfrak{A} = (S, f)$ over the alphabet $A = \bigcup_{i=1}^n A_n$, having the amalgamated sum $S = *_{i=1}^n(S_i, s_i^0)$ of the sets S_i as set of states. The transition function f is given by

$$f(s_i, a_j) = \varphi_{ij}(s_i, a_j) \quad \text{and} \quad f(s_0, a_j) = \varphi_{ij}(s_i^0, a_j)$$

for every i, j, and $a_j \in A_j$.

A simple example of an amalgamation of automata can be obtained as follows: Let $\mathfrak{A} = (S, f)$ be an A^+-automaton with initial state s_0. For every $a \in A$, consider the $\{a\}^+$-automaton $\mathfrak{A}_a = (S, f)$ with initial state s_0. The amalgamation $\mathfrak{B} = *_{a\in A}(\mathfrak{A}_a, \varphi_{a_i a_j})$ with $\varphi_{a_i a_j}(s_i, a_j) = f(s, a_j)$ for every pair $a_i, a_j \in A$ yields an A^+-automaton covering $\mathfrak{A}$ (in fact $\mathfrak{A}$ and $\mathfrak{B}$ have the same transition semigroup).

If each $\mathfrak{A}_i = (S_i, f_i)$ $(i = 1, 2, \ldots, n)$ is the minimal A_i^+-automaton recognizing the languages C_i^+ (C_i^+ = stabilizer of s_i^0 in A_i^+) where each C_i is a complete prefix code, then each $\mathfrak{A}_i$ is a transitive automaton, and the same is true for any amalgamation $\mathfrak{B} = *_{i=1}^n \mathfrak{A}_i$. Hence the stabilizer of s_0 in A^+, with $A = \bigcup_{i=1}^n A_i$, through $\mathfrak{B}$, is itself a complete prefix code. We shall use this process to construct prefix codes C such that Synt(C^+) is a union of groups from elementary codes (i.e. codes C such that Synt(C^+) is completely simple [6]).

3. Domination of team tournaments, and main result

We recall (see [6]) that a team tournament $\mathscr{T}(n, k)$ is a graph composed of k chains

$T_1, T_2, \ldots, T_k$, each chain, called a team, has $n-1$ vertices

$$T_i = \{c_1^i \to c_2^i \to \cdots \to c_{n-1}^i\} \quad (1 \le i \le k),$$

and the arrows between vertices in different teams satisfy the following axioms:

(1) For every $i, 1 \le i \le k$, there is no arrow directed to c_1^i;

(2) For every $i, j, m, i \ne j$ and $m \ne 1$, there exists a unique l, $l \le m$, such that $c_l^i \to c_m^j$ is in $\mathcal{T}(n,k)$;

(3) $\mathcal{T}(n,k)$ has no closed path.

To each tournament $\mathcal{T}(n,k)$ one associates an automaton $\mathfrak{A}\mathcal{T}(n,k)$ on an alphabet $A = \{a_1, a_2, \ldots, a_k\}$ in bijection with the set of teams. We put $\mathfrak{A}\mathcal{T}(n,k) = (S, f)$ with $S = (\bigcup_{i=0}^k T_i) \cup \{0\}$, $f(0, a_i) = c_1^i$, and

$$f(c_l^i, a_j) = \begin{cases} c_m^j & \text{if } c_l^i \to c_m^j \text{ is in } \mathcal{T}(n,k), \\ 0 & \text{otherwise.} \end{cases}$$

In the terminology of Section 2, $\mathfrak{A}\mathcal{T}(n,k)$ is an automaton obtained by amalgamation of k n-cycles using connecting functions $\varphi_{ij} : \{0, 1, \ldots, n-1\} \times \{a_j\} \to \{0, 1, \ldots, n-1\}$ such that

$$\varphi_{ij}(l, a_j) = \begin{cases} m & \text{if } c_l^i \to c_m^j \text{ is in } \mathcal{T}(n,k), \\ 0 & \text{otherwise} \end{cases}$$

(i.e. φ_{ij} induces a non-decreasing permutation on $\{0, 1, \ldots, n-1\}$).

In [6] it was shown that it is possible to define a product of team tournaments by juxtaposition of graphs after insertion of k intermediate points. The product $\mathcal{T}(n,k) \cdot \mathcal{T}(n',k)$ is of the type $\mathcal{T}(n+n',k)$. Furthermore $\mathfrak{A}\mathcal{T}(n,k)$ admits a congruence identifying the states c_l^i and c_m^i whenever $l \ne m$ modulo d, if and only if $n = dq$ and $\mathcal{T}(n,k) = [\mathcal{T}(d,k)]^q$.

Definition 3.1. A team tournament $\mathcal{T}(n,k)$ *dominates* a team tournament $\mathcal{T}(m,l)$ with respect to a factorization of $(1-x^n)/(1-x)$ given by the sequence $k_1 \mid k_2 \mid \cdots \mid k_N \mid n$ if and only if:

(1) $\mathfrak{A}\mathcal{T}(n,k)$ admits a congruence modulo d, where d $(=k_1$ or $k_2)$ is the modulus of the sequence $k_1 \mid k_2 \mid \cdots \mid k_N \mid n$.

(2) m is the product of the consecutive quotients k_2/k_1, $k_4/k_3, \ldots, k_{2i}/k_{2i-1}, \ldots$ with $k_{2i} \le n$.

Referring to the notation used at the beginning of Section 1, let $(1-x^n)/(1-x) = p(x)q(x)$ be the factorization defined by the sequence $k_1 \mid k_2 \mid \cdots \mid k_N \mid n$. This factorization defines a partition π_i of each subset $T_i \cup \{0\}$ of $\mathcal{T}(n,k)$ whose classes are the various sets $C_i(t) = \{c_m^i : m \in A + t\}$, as t runs through the set T of exponents of $q(x)$(A is the set of exponents of $p(x)$). With respect to lower indices of elements of $T_i \cup \{0\}$, π_i is a perfect partition admitting the set $\tau_i = \{c_m^i : m = n - t + 1 \text{ and } t \in T\}$ as a perfect transversal. We call π_i [resp. τ_i] the standard partition [resp. transversal] defined on T_i by the sequence $k_1 \mid k_2 \mid \cdots \mid k_N \mid n$.

In order to construct an amalgamation $\mathfrak{A}\mathcal{T}(n,k)*\mathfrak{A}\mathcal{T}(m,l)$ we need two functions $\varphi: S\times B\to\bar{S}$ and $\psi:\bar{S}\times A\to S$, where $A=\{a_1,a_2,\dots,a_k\}$, $B=\{b_1,b_2,\dots,b_l\}$ are the respective alphabets of $\mathfrak{A}\mathcal{T}(n,k)$ and $\mathfrak{A}\mathcal{T}(m,l)$ and S and $\bar{S}$ their respective set of states. For every pair i,j, $1\le i\le k$, $1\le j\le l$, we consider functions $\beta_{ij}: T_i\cup\{0\}\to\bar{T}_j\cup\{0\}$ and $\alpha_{ji}:\bar{T}_j\cup\{0\}\to T_i\cup\{0\}$ such that $\operatorname{Ker}\beta_{ij}=\pi_i$, β_{ij} onto, and $\operatorname{Im}\alpha_{ji}=\tau_i$, α_{ji} one-to-one, and define φ and ψ by

$$\varphi(c^i_m,b_j)=\beta_{ij}(c^i_m),\qquad \psi(\bar{c}^j_l,a_i)=\alpha_{ji}(\bar{c}^j_l).$$

The second condition on amalgamating functions imposes additional conditions on the functions β_{ij}:

$$\beta_{ij}(0)=\bar{c}^j_1(=\varphi(0,b_j)),\qquad \alpha_{ji}(0)=c^i_1(=\psi(0,a_i)).$$

Any amalgamation $\mathfrak{A}\mathcal{T}(n,k)*\mathfrak{A}\mathcal{T}(m,l)$ constructed as indicated above (where $\mathcal{T}(n,k)$ dominates $\mathcal{T}(m,l)$ with respect to a factorization of $(1-x^n)/(1-x)$ defined by some sequence $k_1\mid k_2\mid\cdots\mid k_N\mid n$) will be called a *standard amalgamation*.

Theorem 3.2. *Any standard amalgamation $\mathfrak{A}\mathcal{T}(n,k)*\mathfrak{A}\mathcal{T}(m,l)$ of two automata of team tournaments $\mathfrak{A}\mathcal{T}(n,k)$ and $\mathfrak{A}\mathcal{T}(m,l)$ is an automaton such that the stabilizer of 0 is C^+ where C is a complete prefix code and the syntactic semigroup of C^+ is a union of groups with one or two $\mathcal{D}$-classes. If the only closed paths in the state graph of $\mathfrak{A}\mathcal{T}(n,k)*\mathfrak{A}\mathcal{T}(m,l)$ are those containing 0, then C is finite. Conversely, any finite prefix code C such that the syntactic semigroup of C^+ is a union of groups with two $\mathcal{D}$-classes is obtainable as the basis of the stabilizer of 0 in a standard amalgamation of two automata of team tournaments.*

Example 3.3. Diagram 3 shows a standard amalgamation $\mathfrak{A}\mathcal{T}(12,1)*\mathfrak{A}\mathcal{T}(4,1)$ with a factorization of $(1-x^{12})/(1-x)$ given by the sequence $1\mid 2\mid 6\mid 12$, i.e.

$$\frac{1-x^{12}}{1-x}=(1+x^2+x^4)(1+x+x^6+x^7).$$

We put $T_1=\{1,2,\dots,11\}$, $\bar{T}_1=\{\bar{1},\bar{2},\bar{3}\}$. The partition π_1 of $T_1\cup\{0\}$ is

$$0\ 2\ 4\mid 1\ 3\ 5\mid 6\ 8\ 10\mid 7\ 9\ 11.$$

The corresponding perfect transversal τ_1 is $12-T+1$ modulo 12 with $T=\{0,1,6,7\}$, i.e. $\tau_1=T$. With $\{a\}$ and $\{b\}$ being the respective alphabets of $\mathfrak{A}\mathcal{T}(12,1)$ and $\mathfrak{A}\mathcal{T}(4,1)$ we have chosen an amalgamation defined by

$$\varphi(0,b)=\varphi(2,b)=\varphi(4,b)=\bar{1},\qquad \psi(0,a)=1,$$

$$\varphi(1,b)=\varphi(3,b)=\varphi(5,b)=\bar{2},\qquad \psi(\bar{1},a)=7,$$

$$\varphi(6,b)=\varphi(8,b)=\varphi(10,b)=\bar{3},\qquad \psi(\bar{2},a)=6,$$

$$\varphi(7,b)=\varphi(9,b)=\varphi(11,b)=0,\qquad \psi(\bar{3},a)=0.$$

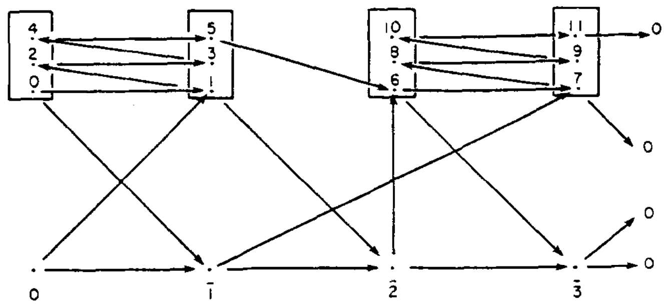

Diagram 3

4. Proofs

The direct part of Theorem 3.2 will follow from the next lemma, where we study perfect transversals on the subsets $T_i \cup \{0\}$ of $\mathfrak{A}\mathcal{T}(n,k)$. Given a factorization of $\mathbb{Z}_n$ defined by $k_1 | k_2 | \cdots | k_N | n$, any transveral of the type $T+s$ where $T = \{0, k_1, 2k_1, \ldots, k_2 - k_1, \ldots\}$ will be referred to as a transversal of type T. Similarly, subsets of $T_i \cup \{0\}$ of the form $\{c_j^i : j \in T+s$ for some $s\}$ will be called transversals of type T in $T_i \cup \{0\}$ (convention: 0 is identified to c_0^i).

Lemma 4.1. *If $\mathfrak{A}\mathcal{T}(n,k)$ admits a congruence modulo d, where d is the modulus of the sequence $k_1 | k_2 | \cdots | k_N | n$, then for every i, j, $1 \le i, j \le k$, any transversal of type T in $T_i \cup \{0\}$ is mapped by a_j onto a transversal of type T in $T_j \cup \{0\}$.*

Proof. For every pair i, j let $\varphi'_{ij} : \mathbb{Z}_n \to \mathbb{Z}_n$ be defined by

$$\varphi'_{ij}(l) = \begin{cases} m-1 & \text{if } c_l^i \to c_m^j \text{ is in } \mathcal{T}(n,k), \\ n-1 & \text{otherwise.} \end{cases} \tag{4.1.1}$$

Viewing $\mathfrak{A}\mathcal{T}(n,k)$ as an amalgamation of k n-cycles the functions φ'_{ij} are related to the connecting functions φ_{ij} by $\varphi'_{ij}(l) = \varphi_{ij}(l, a_j) - 1 \pmod n$. We have $\varphi'_{ij}(0) = 0$ and $\varphi'_{ij}(l) \ge l - 1$ for every l, $0 \le l < n$. Since $\mathfrak{A}\mathcal{T}(n,k)$ admits a congruence modulo d, there is a permutation $\sigma \in \mathcal{S}_d$ such that $\sigma(0) = 0$, $\sigma(l) \ge l - 1$ and $\varphi'_{ij}(i + \lambda d) = \sigma(i) + \lambda d$ for every l, $0 \le l < d$ and for every λ, $0 \le \lambda < n/d$. Hence φ'_{ij} is a perfect permutation of $\mathbb{Z}_n$ mapping every transversal of type T onto a transversal of type T onto a transversal of type T.

Lemma 4.2. *The transition semigroup of a standard amalgamation $\mathfrak{A}\mathcal{T}(n,k) * \mathfrak{A}\mathcal{T}(m,l)$ of two automata of team tournaments is a union of groups with one or two $\mathcal{J}$-classes.*

Proof. A and B being the input alphabets of $\mathfrak{A}\mathcal{T}(n,k)$ and $\mathfrak{A}\mathcal{T}(m,l)$ respectively, in order to prove the lemma it suffices to show that

(1) any word $w \in A^+$ defines a transformation of rank n and having an image which is a cross-section of its kernel;

(2) that any word $w \in (A \cup B)^+$ with at least one occurrence of a letter of B has rank m.

In case $m = n$ it will follow that the transition semigroup of the amalgamation will be completely simple (union of groups with one $\mathscr{J}$-class). In case $m < n$, all the transformations defined by words containing at least one occurrence of a letter of B form the minimal ideal of the amalgamation (transformations of minimal rank), and the transition semigroup has two multiplicatively closed $\mathscr{J}$-class.

The assertion (1) above follows from the fact that all the transformations defined by $w \in A^+$ have their images in $(\bigcup_{i=1}^n T_i) \cup \{0\}$ and we know that the transition semigroup of $\mathfrak{A}\mathscr{T}(n,k)$ is a completely simple semigroup. To prove (2) we proceed by induction on the length of $w \in (A \cup B)^+$. In case $l(w) = 1$, $w = b_j$ for some $b_j \in B$. But $\operatorname{Im} b_j = \bar{T}_j \cup \{0\}$ is a cross-section of $\operatorname{Ker} b_j$ since b_j is a transformation defined by the team tournament $\mathscr{T}(m,l)$.

Assume that all words of length less than the length of w, and containing at least one occurrence of a letter of B, define a partition of type π on each set $T_i \cup \{0\}$, the equality on each set $\bar{T}_j \cup \{0\}$, and have an image which is either a transversal of type T or one of the sets $\bar{T}_j \cup \{0\}$. Let $w = w'z$ with $z \in A \cup B$.

In case $z = a_i$ for some $a_i \in A$, w' has the properties mentioned above. If $\operatorname{Im} w' \subsetneq \operatorname{Im} a_j$ for some $a_j \in A$, then $\operatorname{Im} w'$ is a transversal of type T in $T_j \cup \{0\}$ mapped by a_i on a transversal of type T in $T_i \cup \{0\}$ (Lemma 4.1}. Since w' defines a partition of type π on $T_i \cup \{0\}$, it follows that $\operatorname{Im} w = \operatorname{Im} w'a_i$ is a cross-section of $\operatorname{Ker} w = \operatorname{Ker} w'$. Thus rank w = rank $w' = m$. If $\operatorname{Im} w' = \operatorname{Im} b_j$ for some $b_j \in A$, then $\operatorname{Im} w$ is the perfect transversal τ_i (see the definition of ψ in the amalgamation) in $\operatorname{Im} a_i$, which is a cross-section of the partition defined by w' on $T_i \cup \{0\}$. Hence rank w = rank $w' = m$.

In case $z = b_i$ for some $b_i \in B$, either w' contains at least one letter of B, or $w' \in A^+$. If w' contains a letter of B, then as above $\operatorname{Im} w' \subsetneq \operatorname{Im} a_j$ or $\operatorname{Im} w' = \operatorname{Im} b_j$. In both cases b_i maps $\operatorname{Im} w'$ onto $\bar{T}_i \cup \{0\}$ and rank w = rank $w' = m$. If $w' \in A^+$, we write $w' = a_{i_1} a_{i_2} \cdots a_{i_s}$ with $a_{i_1}, \ldots, a_{i_s} \in A$. By the induction hypothesis, $a_{i_2} \cdots a_{i_s} b_i$ defines a partition π_{i_1} of type π on $T_{i_1} \cup \{0\} = \operatorname{Im} a_{i_1}$ admitting $\operatorname{Im} b_i$ as a cross-section. Since the elements of $\operatorname{Im} b_i$ are mapped by a_{i_1} into distinct classes of π_{i_1}, $\operatorname{Im} b_i$ is a cross section of $\operatorname{Ker} w'b_i = \operatorname{Ker} w$, and rank $w = m$.

In the next series of lemmas we assume that C is a *finite* prefix code on a finite alphabet X and that the syntactic semigroup $\operatorname{Synt}(C^+)$, isomorphic to the transition semigroup of the minimal X^+-automaton $\mathfrak{A}(C^+)$ recognizing C^+, is a union of groups. Since the minimal ideal of $\operatorname{Synt}(C^+)$ is a completely simple semigroup, C is necessarily a complete code and $\mathfrak{A}(C^+)$ is a transitive automaton. We denote by 0 the state of $\mathfrak{A}(C^+)$ whose stabilizer is precisely C^+.

We recall that in a transformation semigroup which is a union of groups, each transformation defines a permutation on its image. In case this transformation semigroup is the syntactic semigroup of C^+ where C is a *finite* code on X, each

$x \in X$ induces a transformation on the set of states of $\mathfrak{A}(C^+)$ which is a cycle containing 0 (see [5], Prop. 1.2, Ch. 8). We shall write

$$x = \begin{pmatrix} S_0 & S_1 & \cdots & S_{n-2} & S_{n-1} \\ 1 & 2 & & n-1 & 0 \end{pmatrix}$$

with $i \in S_i$ for every $i, 0 \le i \le n-1$, $\bigcup_{i=0}^{n-1} S_i$ being the set of states of $\mathfrak{A}(C^+)$, and call x a *cyclic transformation*.

Lemma 4.3. *Let*

$$x = \begin{pmatrix} S_0 & S_1 & \cdots & S_{n-1} \\ 1 & 2 & \cdots & 0 \end{pmatrix} \quad \textit{and} \quad y = \begin{pmatrix} R_0 & R_1 & \cdots & R_{m-1} \\ r_1 & r_2 & \cdots & 0 \end{pmatrix}$$

be two cyclic transformations. If x and y generate a union of groups then the equivalence Ker y [*resp.* Ker x] *restricted to* Im x [*resp.* Im y] *defines a perfect partition of this set.*

Proof. Assume that $\operatorname{Im} x \subseteq R_0 \cup R_{i_1} \cup \cdots \cup R_{i_{t-1}}$ with $\operatorname{Im} x \cap R_{i_j} \neq \emptyset$ for every i_j. We shall prove that the set

$$T = \{0, r_{i_1}x^n, r_{i_2}x^n, \ldots, r_{i_{t-1}}x^n\}$$

is a perfect transversal of the partition defined by y on $\operatorname{Im} x$. Since $y = y^{m+1}$, y^m is an idempotent transformation mapping $0, r_{i_1}, r_{i_2}, \ldots, r_{i_{t-1}}$ onto themselves. Thus $\operatorname{Im} xy^m = \{0, r_{i_1}, r_{i_2}, \ldots, r_{i_{t-1}}\}$, and $T = \operatorname{Im} xy^m x^n$. For every $l \ge 1$, $x \mathcal{L} x^l$ implies $xy^m \mathcal{L} x^l y^m$ hence $\operatorname{Im} xy^m = \operatorname{Im} x^l y^m$ is a cross-section of $\operatorname{Ker} x^l y^m$. Consequently, if $r_{i_j} x^l y = r_{i_k} x^l y$ then with α, β such that $\alpha xy^m = r_{i_j}$, $\beta xy^m = r_{i_k}$ we obtain $\alpha xy^m x^l y = \beta xy^m x^l y$, hence $(\alpha xy^m) x^l y^m = (\beta xy^m) x^l y^m$. Since $\operatorname{Im} xy^m$ is a cross-section of $\operatorname{Ker} x^l y^m$, the last equality implies $\alpha xy^m = \beta xy^m$ or $r_{i_j} = r_{i_k}$. Thus for every integer $l \ge 1$, the elements $0x^l, r_{i_1}x^l, \ldots, r_{i_{t-1}}x^l$ are in different classes of the partition induced by y on $\operatorname{Im} x$, showing that T is a perfect transversal for this partition.

Any partition of the set $\{0, 1, 2, \ldots, n-1\}$ admitting a perfect transversal T is such that for any class Γ of the partition we have $\mathbb{Z}_n = \Gamma \oplus T$. It follows that the cardinality of T (t in the proof of Lemma 4.3) divides n, and all classes have the same size.

Lemma 4.4. *Let $C \subset X^+$ be a finite prefix code such that* $\operatorname{Synt}(C^+)$ *is a union of groups. Then for every $x, y \in X$, viewed as transformations of the minimal automaton $\mathfrak{A}(C^+)$ of C^+, at least one of x or y has a kernel which is the identity when restricted to the image of the other.*

Proof. If $\operatorname{Ker} x$ and $\operatorname{Ker} y$ are not the identity when restricted to $\operatorname{Im} y$ and $\operatorname{Im} x$ respectively, then with the notation of Lemma 4.3 there exists $r_j \in R_j$ and $i \in S_i$ such that $0x = r_j x = 1$ and $0y = iy = r_1$ ($r_j \neq 0$ and $i \neq 0$). It follows that $1x^{i-1}y^j x = 0x^i y^j x =$

$iy^jx = r_jx = 1$, and none of the left factors of $x^{i-1}y^jx$ maps 1 onto 0. Thus if $u \in X^+$ is the shortest word such that $1u = 0$, we have $x(x^{i-1}y^jx)^\lambda u \in C$ for every $\lambda \geq 1$. This contradicts the finiteness of C. Thus at least one of x or y induces the identity on the image of the other.

Corollary 4.5. *If x and y are as in Lemma* 4.3 *and* Ker x *restricted to* Im y *is the identity, then the perfect partition defined by* Ker y *on* Im x *admits a transversal with $m =$* rank y *elements.*

Proof. Since rank $yx =$ rank y we have $yx \mathscr{R} y$. But in a finite union of groups xy and yx are $\mathscr{D}$-related (since $xyxy \mathscr{H} xy$ implies $yx \mathscr{D} xy$). Thus rank $xy =$ rank $yx = m$, and Im x meets all the classes of Ker y. It follows that the transversal T of the proof of Lemma 4.3 has exactly m elements.

In case both Ker x and Ker y induce the identity on Im y and Im x respectively, then the subsemigroup of Synt(C^+) generated by x and y consists of mappings having all the same rank. Since Synt(C^+) is a union of groups, this semigroup is a completely simple semigroup (cf. [1]). More generally, if Y denotes the subset of all the letters of X defining transformations on the set of states of $\mathfrak{A}(C^+)$ such that all have the same rank, then the action of Y on the set $\bigcup_{y \in Y}$ Im y defines a Y^+-automaton; the stabilizer of 0 in this Y^+-automaton is $D^+ = C^+ \cap Y^+$, Synt(D^+) is completely simple hence D is an elementary biprefix code [6]. The main theorem of [6] states that D^+ can be obtained as the stabilizer of 0 in the automaton of a team tournament. Consequently C^+ itself can be obtained by an amalgamation of team tournaments of various lengths. We shall prove that in case Synt(C^+) has two $\mathscr{D}$-classes, C^+ is obtainable by a standard amalgamation of two team tournaments.

Lemma 4.6. *Let $C \subsetneq X^+$ be a finite prefix codes such that* Synt(C^+) *is a union of groups. Assume that the cyclic transformations defined by $x, y \in X$ as in Lemma* 4.3 *are such that* Ker x *restricted to* Im y *is the identity. Define two subsets T and K of* Im x *as follows:*

$$T = \{i \in \text{Im } x : ix \in (\text{Im } y)x\}, \qquad K = \{k \in \text{Im } x : ky = 0y\}.$$

Then the sets K and $\bar{T} = n - T$ (modulo n) *form a factorization of the set* $0, 1, \ldots, n-1$ *(i.e. every a, $0 \leq a \leq n-1$, can be written uniquely $a = k + \bar{t}$ with $k \in K$, $\bar{t} \in \bar{T}$).*

Proof. By Lemma 4.3 and Corollary 4.5, Ker y restricted to Im x defines a perfect partition on Im x having T as perfect transversal and K as the class of 0. By Proposition 5.3 of Chapter 3 in [5], $\mathbb{Z}_n = K \oplus T$, hence $\mathbb{Z}_n = K \oplus \bar{T}$. It remains to show that this last factorization does not need any reduction modulo n. For every $k \in K$, $ky = 0y = r_1$. It follows that for every $i \in T$, $i \neq 0$, we have $k < i$. Indeed, if $k > j$ for some non-zero $j \in T$, let $r_l \in$ Im y such that $r_lx = jx$. Then $x^ky^lx^{k-j}$ is a proper left factor of a word in C. Hence there exists $v \in X^*$ such that $x^ky^lx^{k-j}v \in C$ and since $ky^lx^{k-j}=k$ we also have $x^k(y^lx^{k-j})^\lambda v \in C$ for every $\lambda \geq 1$ contradicting the finiteness of C. Thus $k \leq i$ for every $i \neq 0$, $i \in T$, and the inequality is in fact strict ($k<i$) because $K \cap T=\{0\}$. From $k-i<0$ (for $i \neq 0$) we deduce $0<k+(n-i)<n$, showing that $K \oplus \bar{T}$ is a factorization of the set $\{0,1,\ldots,n-1\}$.

As indicated in Section 1, the factorization $K \oplus \bar{T}$ above is given by a sequence $k_1 \mid k_2 \mid \cdots \mid k_N \mid n$. We now proceed to show that if $x_1, x_2, \ldots, x_l$ have the same rank, and if $y_1, y_2, \ldots, y_k$ of the same rank are such that any x_i induces the identity on $\operatorname{Im} y_j$, then the corresponding factorizations $K_i \oplus \bar{T}_j$ are all the same.

Lemma 4.7. *If $x_1, x_2 \in X$ define two cyclic transformations of the same rank and if* $\operatorname{Ker} x_1$, $\operatorname{Ker} x_2$ *restricted to* $\operatorname{Im} y$ *is the identity, then the two factorizations induced by y on* $\operatorname{Im} x_1$ *and* $\operatorname{Im} x_2$ *are identical. Similarly if $y_1, y_2 \in K$ define two cyclic transformations of the same rank and if* $\operatorname{Ker} x$ *restricted to* $\operatorname{Im} y_1$ *and* $\operatorname{Im} y_2$ *is the identity, then y_1 and y_2 induce identical factorizations on* $\operatorname{Im} x$.

Proof. (a) The factorization induced by y on $\operatorname{Im} x_1$ [resp. $\operatorname{Im} x_2$] corresponds to a polynomial decomposition

$$\frac{1-x^n}{1-x}=p_1(x)q_1(x) \quad [\text{resp.} =p_2(x)q_2(x)]$$

with

$$p_1(x)=1+x^{k_1}+x^{k_2}+\cdots+x^{k_{q-1}}, \qquad q_1(x)=1+x^{t_1}+x^{t_2}+\cdots+x^{t_{m-1}}$$

$$[\text{resp. } p_2(x)=1+x^{\kappa_1}+x^{\kappa_2}+\cdots+x^{\kappa_{\chi-1}},\ q_2(x)=1+x^{\tau_1}+x^{\tau_2}+\cdots+x^{\tau_{\mu-1}}].$$

Writing x_1, x_2, and y as cyclic transformations, we have

$$x_1=\begin{pmatrix} S_0 & S_1 & \cdots & S_{n-1} \\ 1 & 2 & \cdots & 0 \end{pmatrix}, \qquad x_2=\begin{pmatrix} \bar{S}_0 & \bar{S}_1 & \cdots & \bar{S}_{n-1} \\ \bar{1} & \bar{2} & \cdots & 0 \end{pmatrix},$$

$$y=\begin{pmatrix} R_0 & R_1 & \cdots & R_{m-1} \\ r_1 & r_2 & \cdots & 0 \end{pmatrix}$$

with $\{0, k_1, k_2, \ldots, k_{q-1}, \kappa_1, \kappa_2, \ldots, \kappa_{\chi-1}\} \subseteq R_0$, and $0, r_1, r_2, r_{m-1}$ are distributed in the classes $S_0, S_{n-t_{m-1}}, \ldots, S_{n-t_1}$ of $\operatorname{Ker} x_1$, and in the classes $\bar{S}_0, \bar{S}_{n-\tau_{\mu-1}}, \ldots, \bar{S}_{n-\tau_1}$ of $\operatorname{Ker} x_2$.

But x_1x_2 and y generate a union of groups. Thus the class of 0 modulo $\operatorname{Ker} y$ restricted to $\operatorname{Im} x_1x_2 = \operatorname{Im} x_2$ and the transversal on $\operatorname{Ker} x_1$ form a factorization of $\mathbb{Z}_n$. Hence $p_1(x)q_2(x) \equiv 0$ modulo $(1-x^n)/(1-x)$. Similarly, considering x_2x_1 and y we obtain $p_2(x)q_1(x) \equiv 0$ modulo $(1-x^n)/(1-x)$. Thus

$$p_1(x)q_2(x)=k(x)p_1(x)q_1(x) \quad \text{and} \quad p_2(x)q_1(x)=l(x)p_2(x)q_2(x).$$

Since the polynomials $p_i(x), q_i(x)$ have coefficients 0 and 1, this implies $p_1(x)=p_2(x)$ and $q_1(x)=q_2(x)$.

(b) In case y_1, y_2 have the same rank and Ker x is the identity on Im y_1, and Im y_2, a similar proof, using the fact that $y_1 y_2$ and $y_2 y_1$ define perfect partitions on Im x, shows that y_1 and y_2 induce identical factorizations on Im x.

Lemma 4.8. *Assume that $x_1, \ldots, x_2, \ldots, x_k$ define cyclic transformations of the same rank n, and let $\mathfrak{A}\mathcal{T}(n,k)$ be the automaton of the team tournament defined on the set $\bigcup_{i=1}^{k} \mathrm{Im}\, x_i$. Assume that $y \in X$ defines a cyclic transformation such that for every i, Ker x_i is the identity on Im y and the factorization induced by y on Im x_i is given by the sequence $s = k_1 \mid k_2 \cdots \mid k_N \mid n$. If $x_1, x_2, \ldots, x_k, y$ generate a union of groups then $\mathfrak{A}\mathcal{T}(n,k)$ admits a congruence modulo d, where d is the modulus of s.*

Proof. Considering the subtournament of $\mathfrak{A}\mathcal{T}(n,k)$ on the two letters x_1, x_2, (for example) and using the notation of Section 3 we define $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ by $\varphi(0) = 0$, $\varphi(l) = m - 1$ if $c_l^1 \to c_m^2$ is in $\mathcal{T}(n,k)$ and $\varphi(l) = n - 1$ otherwise (cf. 4.1.1). By definition of a tournament φ is a bijection satisfying $\varphi(0) = 0$, $\varphi(l) \geq l - 1$. Furthermore, if T is the transversal of the sequence s, we claim that for every $i, \varphi(T+i)$ is a perfect transversal of the partition defined by s. Indeed, considering $yx_1^i x_2$, we observe that Im yx_1^i is $T+i$. Since x_1, x_2, y generate a union of groups $\varphi(T+i) \subsetneq \mathrm{Im}\, x_2$ is a perfect transversal of the partition defined by Ker y. Thus φ is a perfect permutation of $\mathbb{Z}_n$. By Corollary 1.3, φ is a repetition of a permutation σ on $0, 1, \ldots, d-1$ such that $\sigma(0) = 0$ and $\sigma(i) \geq i - 1$ for $0 < i < d - 1$. Hence $\mathfrak{A}\mathcal{T}(n,k)$ admits a congruence modulo d.

The proof of the converse part of Theorem 3.2 can now be completed. If C is a finite prefix code such that Synt(C^+) is a union of groups with two $\mathcal{J}$-classes, then the set of letters defining transformations of the higher rank and of the lower-rank act as two automata of team tournaments $\mathfrak{A}\mathcal{T}(n,k)$ and $\mathfrak{A}\mathcal{T}(m,l)$ respectively. By Lemmas 4.6, 4.7 and 4.8, $\mathcal{T}(n,k)$ dominates $\mathcal{T}(m,l)$. Lemma 4.6 shows that the automaton $\mathfrak{A}(C^+)$ is obtained by a standard amalgamation $\mathfrak{A}\mathcal{T}(n,k) * \mathfrak{A}\mathcal{T}(m,l)$. Since C is finite the only closed paths in the graph of this amalgamation are those containing 0 (and conversely).

5. Remarks

(a) It has been shown in [4] that all the finite prefix codes C such that Synt(C^+) is a union of groups and has a non-trivial group of units are obtainable from decreasing sequences $s_1 = (1 \mid n) > s_2 > \cdots > s_m$ of chains of divisors of n (where n is the order of the cyclic group of units of Synt(C^+)). The partial order relation on chains is defined as follows: $(k_1 \mid k_2 \mid \cdots \mid k_M \mid n) \geq (l_1 \mid l_2 \mid \cdots \mid l_N \mid n)$ if and only if for every $j, 1 \leq 2j \leq N+1$ there exists $i, 1 \leq 2i \leq M+1$ such that $k_{2i-1} \mid l_{2j-1} \mid l_{2j} \mid k_{2i}$. For example the sequences of divisors of 12,

$$s_1 = (1 \mid 12), \qquad s_2 = (1 \mid 2 \mid 4 \mid 12), \qquad s_3 = (4 \mid 12)$$

satisfy $s_1 > s_2 > s_3$. The corresponding transformations

$$a = (0, 1, \ldots, 11), \qquad b = \left(\begin{array}{cc|cc|cc|cc|cc|cc} 0 & 2 & 1 & 3 & 4 & 6 & 5 & 7 & 8 & 10 & 9 & 11 \\ \multicolumn{2}{c|}{3} & \multicolumn{2}{c|}{4} & \multicolumn{2}{c|}{7} & \multicolumn{2}{c|}{8} & \multicolumn{2}{c|}{11} & \multicolumn{2}{c}{0} \end{array}\right),$$

$$c = \left(\begin{array}{cccc|cccc|cccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \multicolumn{4}{c|}{4} & \multicolumn{4}{c|}{8} & \multicolumn{4}{c}{0} \end{array}\right)$$

generate a transformation monoid Synt(C^+) with 3 completely simple $\mathcal{J}$-classes (C is the basis of the stabilizer of 0).

We conjecture that the results above extend to amalgamations of more than two automata of team tournaments, with appropriate modifications concerning the existence of congruences on these automata.

(b) The codes investigated in [4] were shown to be decomposable (except for group codes of order a prime, or synchronized codes). The wider class explored in the present paper allows to construct non trivial examples of nonsynchronized indecomposable codes as shown by the code C in Example 3.3. It is easy to check that the minimal automaton recognizing C^+ is congruence-free which is equivalent to C being indecomposable ([5] Th. 5.6, Ch. 8). The same code provides also a counterexample to a conjecture of D. Perrin [7], stating that if the degree d of an indecomposable code C is a power of a prime then $a^d \in C$ for every $a \in A$. As pointed out in [8] these types of codes are natural candidates for the study of factorizations (in non-commuting variables) of the type $C - 1 = P(A-1)Q$.

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